Towards an algebraic framework for many-valued conditionals

Tommaso Flaminio[1] Hykel Hosni[2]

[1] Dipartimento di Matematica, Università di Siena,

[2] Centro di Ricerca Matematica E. de Giorgi, Scuola Normale Superiore, Pisa

PROGIC09
Groningen, September 17-19, 2009

Conditional algebras

Motivating question

Motivating question

How can we define subjective conditional probabilities over many-valued events?

Motivating question

Motivating question

How can we define subjective conditional probabilities over many-valued events?

Big question underneath

What sort of object does (logically) qualify as a conditional?

Motivating question

Motivating question

How can we define subjective conditional probabilities over many-valued events?

Big question underneath

What sort of object does (logically) qualify as a conditional?

Proposed solution

Let's do it algebraically

Why bother with many-valued events

- subjective view of probability (betting metaphor)
- realistic bets need not have crisp ‘truth values’ on which bettor and bookmaker are able to agree

Why bother with many-valued events

- subjective view of probability (betting metaphor)
- realistic bets need not have crisp 'truth values' on which bettor and bookmaker are able to agree
  - bets on the women's 800 meters should pay in an inverse proportion to the testosterone levels of the winner

Key ideas

Slogan

Instead of conditional probability, focus on *simple probabilities on conditional events*

Key ideas

Slogan

Instead of conditional probability, focus on *simple probabilities on conditional events*

Functionality constraint

Take the probabilistic evaluations of conditionals as probabilities on conditional algebras of events where the latter are evaluated functionally

Key ideas

Slogan

Instead of conditional probability, focus on *simple probabilities on conditional events*

Functionality constraint

Take the probabilistic evaluations of conditionals as probabilities on conditional algebras of events where the latter are evaluated functionally

Rejection constraint

Say that a conditional $\theta \mid \phi$ is *rejected* if the antecedent is rejected. A conditional is accepted if its antecedent is not rejected.

How did we get here

- Dutch-Book Theorem is generalized to a variety of two-valued non-classical logics
  - J.B. Paris, *A note on the Dutch Book method*. Proceedings of the 2nd International Symposium on Imprecise Probabilities and their Applications, Ithaca, New York.(2001)

How did we get here

- Dutch-Book Theorem is generalized to a variety of two-valued non-classical logics
  - J.B. Paris, *A note on the Dutch Book method*. Proceedings of the 2nd International Symposium on Imprecise Probabilities and their Applications, Ithaca, New York.(2001)
- Dutch-Book Theorem for the Łukasiewicz real-valued logic
  - D. Mundici, *Bookmaking over infinite valued events*. Int. J. of Approximate Reasoning, 46, 223–240, 2006
  - J. Kühr, D. Mundici, *De Finetti theorem and Borel states in* $[0,1]$*-valued algebraic logic*. Int. J. of Approximate Reasoning, **46**(3), 605–616, 2007

How to go conditional?

Classically, just take the law of compound probability:

$$P(A \mid B) = \frac{P(A \wedge B)}{P(B)}$$

provided, of course, that $P(B) \neq 0$

The trouble with MV-logic

Let $F(\dot{})$ be a real-valued valuation function. At least three (good) choices for truth-functional $F_\wedge$.

- $F_\wedge(A, B) = F(A) \times F(B)$
- $F_\wedge(A, B) = \min\{F(A), F(B)\}$
- $F_\wedge(A, B) = \max\{0, F(A) + F(B) - 1\}$

The trouble with MV-logic

Let $F(\dot{})$ be a real-valued valuation function. At least three (good) choices for truth-functional $F_\wedge$.

- $F_\wedge(A, B) = F(A) \times F(B)$
- $F_\wedge(A, B) = \min\{F(A), F(B)\}$
- $F_\wedge(A, B) = \max\{0, F(A) + F(B) - 1\}$

How to choose? (Are there any *good reasons* to prefer any of the above?)

Some like product

- T. Kroupa, *Conditional probability on MV-algebras*. Fuzzy Sets and Systems, 149(2):369-381, 2005.
- F. Montagna, *A notion of coherence for books on conditional events in many-valued logic*. Journal of Logic and Computation, (to appear).

Conditional algebras

The subjectivistic interpretation

Key intuition

$P(\theta \mid \phi)$ *represents* an agent's (rational) degree of belief on θ given ϕ in terms of the agent's disposition to buy (or sell) a suitably defined conditional bet

The subjectivistic interpretation

Key intuition

$P(\theta \mid \phi)$ *represents* an agent's (rational) degree of belief on θ given ϕ in terms of the agent's disposition to buy (or sell) a suitably defined conditional bet

Recall our slogan!

Instead of conditional probability let us focus on *simple probabilities on conditional events*

From conditional assertions to events

[An] affirmation of the form 'if p then q' is commonly felt less as an affirmation of a conditional than as a conditional affirmation of the consequent. If, after we have made such an affirmation, the antecedent turns out true, then we consider ourselves committed to the consequent, and are ready to acknowledge error if it proves false. If, on the other hand, the antecedent turns out to have been false, our conditional affirmation is as if it had never been made.[1]

[1] W.V. Quine. *Methods of Logic*. Harvard University Press, 1959.

What we aim at

- flexibility: many-valued conditional events generalizing the boolean case
- modularity: distinct kinds of conditionals can be formally represented by adding/removing axioms as needed

Conditional events

- we take a conditional $\theta \mid \psi$ as an expression reading

  θ *is the case given that* ψ *is the case.*

- formally, we take the conditional $\theta \mid \psi$ just as a pair of formulae
- we say that a conditional $\theta \mid \psi$ is a *many-valued conditional* if θ and ψ are formulae of a many-valued logic

Conditional events

- we take a conditional $\theta \mid \psi$ as an expression reading

  θ *is the case given that* ψ *is the case.*

- formally, we take the conditional $\theta \mid \psi$ just as a pair of formulae
- we say that a conditional $\theta \mid \psi$ is a *many-valued conditional* if θ and ψ are formulae of a many-valued logic

Example

We are interested in formalizing reasoning based on conditional assertions of the form:

- the occurrence of road accidents increases significantly with wet driving conditions

Conditional algebras

Functionality Constraint

Key semantic idea

A many-valued conditional $\theta \mid \psi$ is evaluated by a pair of functions (f_θ, f_ψ)

Functionality Constraint

Key semantic idea

A many-valued conditional $\theta \mid \psi$ is evaluated by a pair of functions (f_θ, f_ψ)

- each function f_ϕ has as domain a set of possible worlds

Functionality Constraint

Key semantic idea

A many-valued conditional $\theta \mid \psi$ is evaluated by a pair of functions (f_θ, f_ψ)

- each function f_ϕ has as domain a set of possible worlds
- for each possible world x, we call the real number $f_\phi(x)$ the *realization of* ϕ *at world* x

Functionality Constraint

Key semantic idea

A many-valued conditional $\theta \mid \psi$ is evaluated by a pair of functions (f_θ, f_ψ)

- each function f_ϕ has as domain a set of possible worlds
- for each possible world x, we call the real number $f_\phi(x)$ the *realization of ϕ at world x*
- similarly if $\theta \mid \psi$ is a conditional event, we call the the pair $(f_\theta(x), f_\psi(x))$ the *realization of $\theta \mid \psi$ at x*.

FC for MV-algebras

Let us fix $k \in \mathbf{N}$, and let $F(k)$ be the set of all those functions $f : [0,1]^k \to [0,1]$ which are continuous, piecewise linear, and such that each piece has integer coefficients (i.e. k-variate McNaughton functions). Then the algebra

$$\mathcal{F}(k) = (F(k), \oplus, \neg, f_\bot, f_\top)$$

where for every $f, g \in F(k)$, and every $x \in [0,1]^k$

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\},\ (\neg f)(x) = 1 - f(x),\ f_\bot(x) = 0, \text{ and } f_\top(x) = 1,$$

is the free MV-algebra over k-free generators.

FC for MV-algebras

Let us fix $k \in \mathbf{N}$, and let $F(k)$ be the set of all those functions $f : [0,1]^k \to [0,1]$ which are continuous, piecewise linear, and such that each piece has integer coefficients (i.e. k-variate McNaughton functions). Then the algebra

$$\mathcal{F}(k) = (F(k), \oplus, \neg, f_\bot, f_\top)$$

where for every $f, g \in F(k)$, and every $x \in [0,1]^k$

$$(f \oplus g)(x) = \min\{1, f(x) + g(x)\},\ (\neg f)(x) = 1 - f(x),\ f_\bot(x) = 0, \text{ and } f_\top(x) = 1,$$

is the free MV-algebra over k-free generators.

$\mathcal{F}(k)$ is the semantic (algebraic) structure for our many-valued events

Conditional events

The betting metaphor from subjective probability in the boolean case takes conditional events to be three-valued. Thus a bet of the form $\theta \mid \psi$ is:

- won, if ψ and θ are both verified

Conditional events

The betting metaphor from subjective probability in the boolean case takes conditional events to be three-valued. Thus a bet of the form $\theta \mid \psi$ is:

- won, if ψ and θ are both verified
- lost, if ψ is verified, and θ is not

Conditional events

The betting metaphor from subjective probability in the boolean case takes conditional events to be three-valued. Thus a bet of the form $\theta \mid \psi$ is:

- won, if ψ and θ are both verified
- lost, if ψ is verified, and θ is not
- called off, if ψ is not verified

Conditional events

The betting metaphor from subjective probability in the boolean case takes conditional events to be three-valued. Thus a bet of the form $\theta \mid \psi$ is:

- won, if ψ and θ are both verified
- lost, if ψ is verified, and θ is not
- called off, if ψ is not verified

Trievents

De Finetti used to stress the peculiar multi-valuedness of conditional events (in the betting interpretation) by referring to them as *trievents*.[a]

[a] B. de Finetti. *The logic of probability*. Philosophical Studies 77 (1):181-190, 1995.

Conditional MV-events

If we let $\theta \mid \psi$ be a MV-conditional event, we face the following question

Conditional MV-events

If we let $\theta \mid \psi$ be a MV-conditional event, we face the following question

> Under which conditions a bet on $\theta \mid \psi$ is to be called off?

Rejection Constraints (RC)

Given the *functionality constraint*, let x be a possible world in $[0, 1]^k$, then there are two ways in which a many-valued bet on $\theta \mid \psi$ might be called-off at x:

Rejection Constraints (RC)

Given the *functionality constraint*, let x be a possible world in $[0,1]^k$, then there are two ways in which a many-valued bet on $\theta \mid \psi$ might be called-off at x:

1. $f_\psi(x) = 0$ (that is if ψ is evaluated by x into 0)

Rejection Constraints (RC)

Given the *functionality constraint*, let x be a possible world in $[0, 1]^k$, then there are two ways in which a many-valued bet on $\theta \mid \psi$ might be called-off at x:

1. $f_\psi(x) = 0$ (that is if ψ is evaluated by x into 0)
2. $f_\psi(x) < 1$ (that is if ψ not evaluated by x into 1)

Rejection Constraints (RC)

Given the *functionality constraint*, let x be a possible world in $[0,1]^k$, then there are two ways in which a many-valued bet on $\theta \mid \psi$ might be called-off at x:

1. $f_\psi(x) = 0$ (that is if ψ is evaluated by x into 0)
2. $f_\psi(x) < 1$ (that is if ψ not evaluated by x into 1)

Observation

Clearly 1 and 2 above coincide in the boolean case

Three constraints

Case 1: Let $\theta \mid \psi$ be a bet such that and consider all those possible worlds $x \in [0,1]^k$ such that $f_\psi(x) > 0$. Then we distinguish two cases:

Three constraints

Case 1: Let $\theta \mid \psi$ be a bet such that and consider all those possible worlds $x \in [0,1]^k$ such that $f_\psi(x) > 0$. Then we distinguish two cases:

(RC_1) **boolean**: The conditional is accepted because it is not rejected. In this case we are only interested in the qualitative acceptance of the conditional $\theta \mid \psi$

Three constraints

Case 1: Let $\theta \mid \psi$ be a bet such that and consider all those possible worlds $x \in [0,1]^k$ such that $f_\psi(x) > 0$. Then we distinguish two cases:

(RC_1) **boolean**: The conditional is accepted because it is not rejected. In this case we are only interested in the qualitative acceptance of the conditional $\theta \mid \psi$

(RC_2) **many-valued**: In this second case, the degree α to which ψ is true at x matters (i.e. $f_\psi(x) = \alpha$ with $0 < \alpha \leq 1$)

Three constraints

Case 1: Let $\theta \mid \psi$ be a bet such that and consider all those possible worlds $x \in [0,1]^k$ such that $f_\psi(x) > 0$. Then we distinguish two cases:

(RC_1) **boolean**: The conditional is accepted because it is not rejected. In this case we are only interested in the qualitative acceptance of the conditional $\theta \mid \psi$

(RC_2) **many-valued**: In this second case, the degree α to which ψ is true at x matters (i.e. $f_\psi(x) = \alpha$ with $0 < \alpha \leq 1$)

Case 2:

(RC_3) Not rejected are just those conditional events whose antecedent ψ is evaluated into 1.

Conditional algebras

Conditional MV-algebras

A conditional MV-algebra is defined as follows:

Conditional MV-algebras

A conditional MV-algebra is defined as follows:

(1) Let $\mathcal{F}(k)$ be a free MV-algebra and $\mathcal{F}(k)^{\perp}$ its MV-bunch

(2) Let $\mathcal{F}(k \times k^{\perp})$ be the MV-algebra freely generated by the pairs (f, g) in the cartesian product $\mathcal{F}(k) \times \mathcal{F}(k)^{\perp}$

(3) For $j = 1, 2, 3$, the conditional MV-algebra $\mathcal{F}(k) \mid_j \mathcal{F}(k)^{\perp}$ is defined as the quotient

$$\mathcal{F}(k \times k^{\perp})/\mathfrak{I}_j$$

where $\mathfrak{I}_j$ is a suitably defined MV-ideal satisfying the rejection constraint (RC_i)

Conditional algebras

1. Introduction and Motivation
2. Back to square one
3. Constraints for MV-conditionals
4. Conditional MV-algebras
5. States on conditional MV-algebras

$j = 1$

For $j = 1$, we claim that the algebra $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^{\perp}$ is a good semantics for those conditionals $\theta \mid \psi$ which are which are rejected in all those worlds x such that$f_{\psi}(x) = 0$.

$j = 1$

For $j = 1$, we claim that the algebra $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^{\perp}$ is a good semantics for those conditionals $\theta \mid \psi$ which are which are rejected in all those worlds x such that$f_{\psi}(x) = 0$. Recall from RC_1 that the evaluation of ψ needs to be *crisp*: if $f_{\psi}(x) > 0$, then the bet is accepted and rejected otherwise.

$j = 1$

For $j = 1$, we claim that the algebra $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^{\perp}$ is a good semantics for those conditionals $\theta \mid \psi$ which are which are rejected in all those worlds x such that$f_{\psi}(x) = 0$. Recall from RC_1 that the evaluation of ψ needs to be *crisp*: if $f_{\psi}(x) > 0$, then the bet is accepted and rejected otherwise.

The Conditional MV-algebra $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^{\perp}$ is the quotient algebra $\mathcal{F}(k \times k^{\perp})/\mathfrak{I}_1$, where $\mathfrak{I}_1$ is the ideal generated by the following elements:

(1) $d((\top, g), \top)$ for all $g \in \mathcal{F}(k)^{\perp}$, such that $\ker(g) = \emptyset$.

(2) $d((f, g), (f, g'))$ for all $g, g' \in \mathcal{F}(k)^{\perp}$ such that $\ker(g) = \ker(g')$,

(3) $d((f \star f', g), (f, g) \star (f', g))$ for $\star \in \{\wedge, \vee, \oplus\}$,

(4) $d((\neg f, g), \neg(f, g))$.

Bayes states

Idea

Is a state on a conditional MV-algebra a conditional state ?

Bayes states

Idea

Is a state on a conditional MV-algebra a conditional state ?

Theorem. Let s be a *state* on $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^{\perp}$, that is $s : \mathcal{F}(k) \mid_1 \mathcal{F}(k)^{\perp} \longrightarrow [0,1]$ is such that:

- $s(\top) = 1$ and,
- for all $f_1 \mid g_1, f_2 \mid g_2 \in \mathcal{F}(k) \mid \mathcal{F}(k)^{\perp}$, if $(f_1 \mid g_1) \odot (f_2 \mid g_2) = \perp$, then $s((f_1 \mid g_1) \oplus (f_2 \mid g_2)) = s(f_1 \mid g_1) + s(f_2 \mid g_2)$.

Then s satisfies the following properties:

- $s(\cdot \mid g)$ is a state on $\mathcal{F}(k)$ for all $g \in \mathcal{F}(k)^{\perp}$ with $\ker(g) = \emptyset$.

Bayes states

Idea

Is a state on a conditional MV-algebra a conditional state ?

Theorem. Let s be a *state* on $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^\perp$, that is $s : \mathcal{F}(k) \mid_1 \mathcal{F}(k)^\perp \longrightarrow [0,1]$ is such that:

- $s(\top) = 1$ and,
- for all $f_1 \mid g_1, f_2 \mid g_2 \in \mathcal{F}(k) \mid \mathcal{F}(k)^\perp$, if $(f_1 \mid g_1) \odot (f_2 \mid g_2) = \perp$, then $s((f_1 \mid g_1) \oplus (f_2 \mid g_2)) = s(f_1 \mid g_1) + s(f_2 \mid g_2)$.

Then s satisfies the following properties:

- $s(\cdot \mid g)$ is a state on $\mathcal{F}(k)$ for all $g \in \mathcal{F}(k)^\perp$ with $\ker(g) = \emptyset$.
- $s(g \mid g) = 1$ for every $g \in B(\mathcal{F}(k)) \cap \mathcal{F}(k)^\perp$.

Bayes states

Idea

Is a state on a conditional MV-algebra a conditional state ?

Theorem. Let s be a *state* on $\mathcal{F}(k) \mid_1 \mathcal{F}(k)^\perp$, that is $s : \mathcal{F}(k) \mid_1 \mathcal{F}(k)^\perp \to [0, 1]$ is such that:

- $s(\top) = 1$ and,
- for all $f_1 \mid g_1, f_2 \mid g_2 \in \mathcal{F}(k) \mid \mathcal{F}(k)^\perp$, if $(f_1 \mid g_1) \odot (f_2 \mid g_2) = \perp$, then $s((f_1 \mid g_1) \oplus (f_2 \mid g_2)) = s(f_1 \mid g_1) + s(f_2 \mid g_2)$.

Then s satisfies the following properties:

- $s(\cdot \mid g)$ is a state on $\mathcal{F}(k)$ for all $g \in \mathcal{F}(k)^\perp$ with $\ker(g) = \emptyset$.
- $s(g \mid g) = 1$ for every $g \in B(\mathcal{F}(k)) \cap \mathcal{F}(k)^\perp$.
- $s(f \odot f' \mid g) = s(f' \mid g)s(f \mid f' \odot g)$ for any $f \in \mathcal{F}(k)$, $f' \in B(\mathcal{F}(k))$, $g \in B(\mathcal{F}(k)) \cap \mathcal{F}(k)^\perp$ such that $f' \odot g \in \mathcal{F}(k)^\perp$

Key results

Within this framework we are able to prove that

- conditional MV-algebras are a proper generalization of conditional boolean algebras
- conditional states (in the sense of Gerla) are representable in a suitable conditional MV-algebra